Mathematisches Institut — Universität zu Köln
Prof. Dr. Sabatini — 15.05.2019

Algebraic Topology – Homework 7

Due date : May 22th in class.

Exercise 25. (6 Points)

Find a CW-complex structure on $\mathbb{CP}^n$ with only one cell of dimension $2i$ for all $i = 0, \ldots, n$. Describe the attaching maps $\varphi_i : \partial D^{2i} \longrightarrow X^{2i-1}$ explicitly.

Exercise 26. (6+6 Points)

Let X be the topological space obtained as a quotient of $\mathbb{C}^2 \backslash \{0\}$ via the equivalence relation

$$(z, w) \sim (\lambda z, \lambda^3 w) \quad \text{for all } \lambda \in \mathbb{C} \backslash \{0\} .$$

(i) Endow X with a CW-structure and describe the attaching maps $\varphi_\alpha : \partial D_\alpha \longrightarrow X^{n-1}$ for all α, n.

(ii) Compute the (cellular) homology groups of X.

Exercise 27. (6 Points)

Let X be a topological space which can be endowed with the structure of a finite CW-complex $X = X^n$, i.e., the number of i-dimensional cells, denoted by c_i, is finite for every i, and zero for $i > n$. Define the **Euler characteristic** of X to be

$$\chi(X) = \sum_{i=0}^{n} (-1)^i c_i .$$

Prove that $\chi(X)$ only depends on the homotopy type of X.

Exercise 28. (8 Points)

Regard S^n as the quotient of two n-simplices $\triangle_1^n$ and $\triangle_2^n$ identified along the boundary as explained in class. One can show that the identity map

$$\mathrm{Id}_i : \triangle_i^n \longrightarrow \triangle_i^n$$

is a cycle that generates $H_n(\triangle_i^n, \partial \triangle_i^n)$, $i \in \{1, 2\}$. Using this fact, prove that the class represented by the difference $\triangle_1^n - \triangle_2^n$ is a generator of $\widetilde{H}_n(S^n)$.

Exercise 29. (6+6+6 Points)

Let X be a finite $\triangle$-complex. By viewing an n-simplex of X as its characteristic map $\sigma_\alpha : \triangle^n \longrightarrow X$, we obtain a canonical homomorphism $H_n^\triangle(X) \longrightarrow H_n(X)$.
Just like in singular homology, one can define relative simplicial homology groups of the pair (X, A), where A is a $\triangle$-subcomplex of X, as the homology groups with chains given by $\triangle_n(X)/\triangle_n(A)$. Then the Snake Lemma implies the existence of a long exact sequence of simplicial homology groups for the pair (X, A)

$$\ldots \longrightarrow H_n^\triangle(A) \longrightarrow H_n^\triangle(X) \longrightarrow H_n^\triangle(X, A) \longrightarrow H_{n-1}^\triangle(A) \longrightarrow H_{n-1}^\triangle(X) \longrightarrow \ldots$$

(i) Let X^k be the k-dimensional skeleton of X, i.e, the union of all the simplices of X of dimension less or equal to k. Use the definition of relative simplicial homology groups to prove that $H_n^\triangle(X^k, X^{k-1})$ is zero if $n \neq k$ and is a free abelian group generated by the n-simplices of X if $n = k$.
(ii) Use the fact that the identity map $\mathrm{Id} : \triangle^n \longrightarrow \triangle^n$ generates $H_n(\triangle^n, \partial\triangle^n)$ to prove that the canonical homomorphism $H_n^\triangle(X^k, X^{k-1}) \longrightarrow H_n(X^k, X^{k-1})$ is an isomorphism for all n.
(iii) Use the long exact sequence of the pair (X^k, X^{k-1}) to prove that

$$H_n^\triangle(X^k) \cong H_n(X^k)$$

for all n, k, and hence $H_n^\triangle(X) \cong H_n(X)$. *Hint :* for each fixed n, use induction on k.