Algebraic Topology – Homework 1

Due date : April 10th in class

Exercise 1. (8+8 Points)

Let X be a topological spaced endowed with a Δ-structure given by a collection of continuous maps $\sigma_\alpha : \Delta^n \longrightarrow X$ where n depends on the index α and Δ^n denotes the standard n-dimensional simplex.

(1) Prove that X is a Hausdorff space.
(2) Let $\mathring{\Delta}^n$ denote the interior of Δ^n. Show that

$$\sigma|_{\mathring{\Delta}^n} : \mathring{\Delta}^n \longrightarrow X$$

is a homeomorphism onto its image.

Exercise 2. (8+8+8 Points)

(1) Recall that the **wedge sum** $X \vee Y$ of two topological spaces X and Y, with given points $x \in X$ and $y \in Y$, is the quotient of the disjoint union $X \amalg Y$ obtained by identifying x with y. (Here you *cannot* use Proposition 1 in Exercise 4, you need to do it directly.)

Endow the wedge of g circles with the "obvious" Δ-complex structure (which has exactly g 1-simplices) and compute its simplicial homology.

(2) Endow S^2 with a Δ-complex structure with two Δ^2 simplices identified along their boundaries with identity map. Compute the simplicial homology of S^2.
(3) Endow the Klein bottle with a Δ-complex structure and compute its simplicial homology groups.

Exercise 3. (10 Points)

Let X_n be the topological space obtained from an n-gon with identifications on the boundary induced by the word $\overbrace{a \cdot a \cdot \cdots \cdot a}^{n \text{ times}}$. (For example, X_2 is $\mathbb{R}P^2$.) This space is also called **n-fold dunce cap**.
Endow X_n with a suitable Δ-complex structure, and compute its simplicial homology groups. (Hint : Consider the regular polygon with n edges. Consider the Δ-complex structure obtained by adding a vertex in the middle of it, and 1-simplices pointing radially inward.)