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Symplectic Geometry – Homework 7

Due on June 8th 2015, in class

Exercise 1.

This is a follow-up/prequel to Exercise 3 of Homework 6. A Lie group $(G, \circ)$ is a smooth manifold endowed with a group operation $\circ$ such that the multiplication map $\circ : G \times G \to G$ (defined by $(x, y) \longmapsto x \circ y$) and the inverse map $i : G \to G$ (defined by $g \longmapsto g^{-1}$) are smooth.

- Argue in maximally two lines that $GL(2n, \mathbb{R})$ is a Lie group.

One fact that one can prove is Cartan's somewhat surprising Closed Subgroup theorem.

Theorem 1. *Let G be a Lie group and $H \subseteq G$ a subgroup that is also a closed subset of G. Then H is an embedded Lie subgroup.*

This means that any subgroup that is topologically closed is automatically a submanifold, and the induced multiplication and inversion are smooth!

- Prove that $Sp(2n, \mathbb{R}), GL(n, \mathbb{C}), U(n), O(2n, \mathbb{R}), SO(2n, \mathbb{R})$ are Lie groups.

In the previous exercise we have shown that $T_{\mathbb{1}}SP(2n, \mathbb{R}) \subseteq \mathfrak{sp}(2n, \mathbb{R})$, where

$$\mathfrak{sp}(2n, \mathbb{R}) = \{X \in M_{2n \times 2n}(\mathbb{R}) \mid X^T J_0 + J_0 X = 0\}$$

Recall that for a matrix $X \in M_{2n \times 2n}(\mathbb{R})$ the matrix exponential is given by

$$\exp(X) = \sum_{k=0}^{\infty} \frac{1}{k!} X^k.$$

- Show that for $X \in \mathfrak{sp}(2n, R)$ and $t \in \mathbb{R}$ one has $\psi_t(X) := \exp(tX) \in Sp(2n, \mathbb{R})$, $\psi_0(X) = \mathbb{1}$ and $\frac{d}{dt}\big|_{t=0} \psi_t(X) = X$. So we can identify the tangent space at the identity with $\mathfrak{sp}(2n, \mathbb{R})$.
- Calculate the dimension of $Sp(2n, \mathbb{R})$.
- Let $X, Y \in \mathfrak{sp}(2n, \mathbb{R})$. Show that that $[X, Y] \in \mathfrak{sp}(2n, \mathbb{R})$ where $[\cdot, \cdot]$ is the ordinary commutator bracket of matrices.

Thus $\mathfrak{sp}(2n, R)$ is a *Lie subalgebra* of $M_{2n \times 2n}(\mathbb{R})$. This lie algebra can be related to the Lie bracket of vector fields of so called *left invariant vector fields* of the group $Sp(2n, \mathbb{R})$. Much of the structure of Lie groups is related to properties of its Lie algebra, and (simple) Lie groups can actually be classified by their Lie algebras.

<u>Exercise</u> 2.

Let M be an orientable surface.

- Let ω_0, ω_1 be two symplectic forms on M inducing the same orientation on M. Show that
$$\omega := (1-t)\omega_0 + t\omega_1$$
is symplectic for all $t \in [0,1]$.
- Suppose $[\omega_0] = [\omega_1] \in H^2_{\text{deRham}}(M)$ and M is closed. Prove that there exists a one-parameter family of diffeomorphisms $\phi_t : M \to M$ such that $\phi_0 = id$ and $\phi_1^* \omega_1 = \omega_0$.

<u>Exercise</u> 3.

Find two symplectic forms ω_0, ω_1 on $\mathbb{R}^4$ that induce the same orientation on $\mathbb{R}^4$ such that

$$\omega := (1-t)\omega_0 + t\omega_1$$

is not symplectic for some $t \in [0,1]$. Find a path of symplectic forms that connects ω_0 and ω_1.

<u>Exercise</u> 4.

One can prove that a non-zero one form α on a two dimensional manifold M can locally be written as $\alpha = f dg$, with $f, g : M \to \mathbb{R}$ functions. Prove, using this fact, the two dimensional Darboux theorem :

Theorem 2. *Let (M, ω) be a sympletic two dimensional manifold. For each $p \in M$ there exists a coordinate chart $(U \ni p, \phi : U \to \mathbb{R}^2)$ such that*

$$(\phi^{-1})^* \omega = dx \wedge dy.$$

Hint : Locally ω has a primitive α as ω is closed. What does the non-degeneracy of ω imply for this primitive ?