Mathematisches Institut
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Algebraic Topology – Homework 4

Due date : May 2nd during the Exercise Session with Dr. Alexander Caviedes or at 11 o'clock in the office 005a of the Math Institute.

Exercise 12. (4 Points)

Show that for a pair of spaces (X, A), the inclusion $A \hookrightarrow X$ induces isomorphisms on all homology groups if and only if $H_n(X, A) = \{0\}$ for all n.

Exercise 13. (6+6 Points)

(i) Show that $H_0(X, A) = 0$ if and only if A meets each path-component of X.
(ii) Show that $H_1(X, A) = 0$ if and only if $H_1(A) \longrightarrow H_1(X)$ is surjective and each path-component of X contains at most one path-component of A.

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A **strong deformation retract** of a space X onto a subspace A is a continuous map $F : X \times [0, 1] \longrightarrow X$, such that $F(x, 0) = x$ and $F(x, 1) \in A$ for all $x \in X$, and $F(a, t) = a$ for all $a \in A$ and $t \in [0, 1]$.

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Exercise 14. (8+8 Points)

In this Exercise you will need to assume the following Proposition :

Proposition. *Suppose that a non-empty closed set $A \subset X$ admits a neighborhood V in X such that V strongly deformation retracts onto A. Then the quotient map*

$$q : (X, A) \longrightarrow (X/A, A/A)$$

induces an isomorphism in homology, hence

$$H_k(X, A) \cong H_k(X/A, A/A) \cong \widetilde{H}_k(X/A)$$

for all nonnegative integers k.

(i) Show that $\widetilde{H}_k(X) \cong \widetilde{H}_{k+1}(SX)$ for all nonnegative integer k, where SX denotes the suspension of X.

(ii) Using (i), compute the reduced homology groups of the n-dimensional sphere S^n given the fact that

$$\widetilde{H}_k(S^1) \cong \begin{cases} \mathbb{Z} & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases} .$$

<u>Exercise</u> 15. (6+6+6 Points)

(i) Show that if A is a retract of X (i.e. there exists a continuous map $r\colon X \longrightarrow A$ which is the identity on $A \subseteq X$) then the map $H_k(X) \longrightarrow H_k(A)$ induced by the inclusion $A \subset X$ is injective.

(ii) Let $D^{n+1} = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \ldots + x_{n+1}^2 \leq 1\}$ be the closed unit disc in $\mathbb{R}^{n+1}$ and $S^n \subset D^{n+1}$ be the n-dimensional unit sphere seen as a subspace of D^{n+1}. Is S^n a retract of D^{n+1}?

(iii) Let $\mathbb{RP}^2$ be the 2-dimensional real projective space obtained from S^2 by identifying antipodal points. Let $S^1 = \{(x_1, x_2, 0) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = 1\}$ be the unit circle seen as a subspace of S^2 and let $\mathbb{RP}^1$ be the 1-dimensional real projective space obtained from S^1 by identifying antipodal points and seen as a subspace of $\mathbb{RP}^2$. Is $\mathbb{RP}^1$ a retract of $\mathbb{RP}^2$?