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Symplectic Geometry – Homework 3

Due on May 4th 2015, in class

Exercise 1.

Construct explicit trivializations of $TS^3 \cong S^3 \times \mathbb{R}^3$ and $TS^7 \cong S^7 \times \mathbb{R}^7$.
(*Hint* : $S^3 \subseteq \mathbb{H}, \ldots$)

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Let $\{g_{\alpha\beta}\}$ and $\{\widetilde{g}_{\alpha\beta}\}$ be two sets of smooth functions, each defined from $U_\alpha \cap U_\beta \longrightarrow GL(k, F)$ (where F is either $\mathbb{R}$ or $\mathbb{C}$) for every $U_\alpha \cap U_\beta \neq \emptyset$, for some open cover $\{U_\alpha\}$ of M.

Definition 1. *We say that the two sets of functions $\{g_{\alpha\beta}\}$ and $\{\widetilde{g}_{\alpha\beta}\}$ are* **equivalent** *if for every α there exist smooth maps $\lambda_\alpha \colon U_\alpha \longrightarrow GL(k, F)$ such that*

$$g_{\alpha\beta} = \lambda_\alpha \cdot \widetilde{g}_{\alpha\beta} \cdot \lambda_\beta^{-1}$$

for every α, β such that $U_\alpha \cap U_\beta \neq \emptyset$. (The symbol $\cdot$ above denotes matrix multiplication.)

Exercise 2.

- Given vector bundles E and $\widetilde{E}$ over M, with transition functions $\{g_{\alpha\beta}\}$ and $\{\widetilde{g}_{\alpha\beta}\}$ for some common open cover $\{U_\alpha\}$, construct explicitly transition functions for the vector bundles $E \oplus \widetilde{E}$ and $E \otimes \widetilde{E}$.
- Let $\pi \colon E \longrightarrow M$ and $\widetilde{\pi} \colon \widetilde{E} \longrightarrow M$ be two vector bundles that trivialise over the same open cover U_α. Prove that following

  Theorem 2. *Two bundles as above are isomorphic if and only if their transition functions $\{g_{\alpha\beta}\}$ and $\{\widetilde{g}_{\alpha\beta}\}$ are equivalent.*

- Suppose that E is line bundle (over $\mathbb{R}$ or over $\mathbb{C}$), and consider its dual E^*. Prove that $E \otimes E^*$ is trivial. Can you exhibit a nowhere zero section ?

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Definition 3. *A* **Euclidean metric** g *on a vector bundle* E *is a family of inner products* $g_x : E_x \times E_x \to \mathbb{R}$, $x \in M$ *that varies smoothly with the basepoint. That is, for all sections* s, s' *of* E *we have that* $x \longmapsto g_x(s(x), s'(x))$ *is a smooth function on* M.

Exercise 3.

- Show that the set of real line bundles over M forms an abelian group with respect to the tensor product.
- Show that if a real vector bundle E possesses a Euclidean metric, then E is isomorphic to E^*.
- Show that if a real line bundle E possesses a Euclidean metric then E represents an element of order ≤ 2 in the group described above.
- What does this mean for the Möbius strip ?

Remark : a Euclidean metric on the tangent bundle of a manifold is called a *Riemannian metric*. It is a fact that if M is paracompact (we will always assume this), then every vector bundle over it can be equipped with a Euclidean metric.

Exercise 4.

Recall that $\mathbb{CP}^1$ is the space of complex lines in $\mathbb{C}^2$, i.e. $\mathbb{CP}^1 = \mathbb{C}^2 \setminus \{0\}/\sim$, where

$$(z, w) \sim (z', w') \Leftrightarrow (z, w) = \lambda(z', w') \quad \text{for some} \quad \lambda \in \mathbb{C}^*$$

Define the set

$$H = \{([z, w], (a, b)) \in \mathbb{CP}^1 \times \mathbb{C}^2 \mid (a, b) = \lambda(z, w) \quad \text{for} \quad \lambda \in \mathbb{C}\}$$

Recall that $\mathbb{CP}^1$ is covered by two charts $U_1 = \{[z, 1] \in \mathbb{CP}^1\}$ and $U_2 = \{[1, w] \in \mathbb{CP}^1\}$. Show that H is a complex line bundle by constructing transition functions for H with respect to this cover.