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Symplectic Geometry – Homework 1

Due on April 20th 2015, in class

Recall that given a vector space V over $\mathbb{R}$, with $\dim(V) = m$ and $0 < m < \infty$, an **orientation** on V is defined as follows : Let $\mathcal{B}$ be the set of bases of V, and introduce the equivalence relation $\sim$ on $\mathcal{B}$ which declares two bases $\underline{u}$ and $\underline{v}$ equivalent if $\det(A) > 0$, where $\underline{v} = A\underline{u}$. Then the set of equivalence classes $\mathcal{B}/\sim$ consists of two elements B_0 and B_1, and picking an orientation on V amounts to choosing one of these two elements, say B_0. Hence a basis $\underline{v}$ is *positively* oriented if $\underline{v} \in B_0$.

Exercise 1.

- Observe that $\Lambda^m(V^*) = \mathbb{R}$, hence $\Lambda^m(V^*) \setminus \{0\}$ has two connected components (with respect to the standard topology on $\mathbb{R}$). Explain in which sense picking an orientation on V is equivalent to choosing one of these two connected components.
- Let V be as above, with $m = 2n$, and let $\omega \in \Lambda^2(V^*)$. Prove that ω is *symplectic* if and only if $\omega^n \neq 0$.

  Thus *any symplectic vector space comes with a 'natural orientation', i.e. that determined by (the connected component of* $\Lambda^{2n}(V^*) \setminus \{0\}$ *containing)* ω^n.

Exercise 2.

Exercises 2, 3, 4, 5, 6, 8, 9 on page 8 of Cannas da Silva book, available online on my webpage.

Exercise 3.

Let $A \in \mathrm{Sp}(2n)$. From the definition of $\mathrm{Sp}(2n)$ check that $\det(A)^2 = 1$. Is $\det(A) = 1$?