Mathematisches Institut | Universität zu Köln
Prof. Dr. Sabatini | 10.04.2019

Algebraic Topology – Homework 2

Due date : April 17th in class

Exercise 4. (15 Points)

In this exercise you will need the following

Proposition 1. *Let X_α be topological spaces endowed with a Δ-complex structure, and consider $\vee_\alpha X_\alpha$, which can also be endowed with a Δ-complex structure. Assume that each of the point $x_\alpha \in X_\alpha$, identified in the wedge sum $\vee_\alpha X_\alpha$, has a contractible neighborhood in X_α. Then $H_i^\Delta(\vee_\alpha X_\alpha) \cong \bigoplus_\alpha H_i^\Delta(X_\alpha)$ for every $i > 0$.*

Given finitely generated abelian groups G_1 and G_2, with G_2 free, describe a finite 2-dimensional Δ-complex X which is connected and such that $H_1^\Delta(X) \cong G_1$ and $H_2^\Delta(X) \cong G_2$. (Hint : use the *fundamental theorem of finitely generated abelian groups*).

Exercise 5. (10 Points)

Let X be a nonempty topological space with $n < \infty$ path-connected components. Prove that $\widetilde{H}_0(X) \simeq \mathbb{Z}^{n-1}$ if $n > 1$ and $\widetilde{H}_0(X) = 0$ if $n = 1$ explicitly, by exhibiting a basis of it.

Exercise 6. (10 Points)

Let X, Y be topological spaces, and $f\colon X \longrightarrow Y$ a constant map. Prove that $f_*\colon H_i(X) \longrightarrow H_i(Y)$ is the zero homomorphism for every $i > 0$.

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Let $\{A_n\}_{n\in\mathbb{Z}}$ be a sequence of abelian groups, and $\{\alpha_n\colon A_{n+1} \longrightarrow A_n\}_{n\in\mathbb{Z}}$ be homomorphisms

$$\cdots \longrightarrow A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \longrightarrow \cdots$$

so that $\operatorname{Ker} \alpha_n = \operatorname{Im} \alpha_{n+1}$ for every n. Thus the pair $(A_*, \alpha_*) = \{(A_n, \alpha_n)\}_{n\in\mathbb{Z}}$ is a chain complex with *trivial homology*, and is called an **exact sequence**.
In particular

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

is called a **short exact sequence**. This is equivalent to saying that α is *injective*, β is *surjective* and $\operatorname{Im}\alpha = \operatorname{Ker}\beta$, thus implying that $B/\operatorname{Im}\alpha \simeq C$.

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<u>Exercise</u> 7. (15 Points)

Suppose that

$$\cdots \longrightarrow A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \longrightarrow \cdots$$

is an exact sequence. Prove that for every n there is a short exact sequence of the form

$$0 \longrightarrow \operatorname{Coker} \alpha_{n+2} \xrightarrow{\widetilde{\alpha}_{n+1}} A_n \xrightarrow{\alpha'_n} \operatorname{Ker} \alpha_{n-1} \longrightarrow 0$$

where $\operatorname{Coker} \alpha_i$ denotes the cokernel of α_i, i.e. it is equal to $A_{i-1}/\operatorname{Im} \alpha_i$. Note that you need to define the maps $\widetilde{\alpha}_{n+1}$ and α'_n, and prove that they are well-defined.