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Symplectic Geometry – Homework 9

Due on June 22th 2015, in class

Exercise 1.

Let (M, g) be a Riemannian manifold. The metric g induces a metric $\tilde{g}$ on T^*M. It is defined pointwise, for $q \in M, p_1, p_2 \in T_q^*M$, by

$$\tilde{g}_q(p_1, p_2) = g(v_1, v_2) \quad \text{where} \quad v_1, v_2 \in T_qM \quad \text{such that} \quad g_q(v_1, \cdot) = p_1 \text{ and } g_q(v_2, \cdot) = p_2.$$

The metric $\tilde{g}$ defines the Hamiltonian $H : T^*M \to \mathbb{R}$

$$H(q, p) = \frac{1}{2}\tilde{g}_q(p, p).$$

- Give an explicit formula of H in local coordinates on T^*M in terms of the metric tensor.
- Compute the Hamiltonian vector field X_H in these coordinates with respect to the canonical symplectic form ω_0 on T^*M.
- Show that if $\gamma : I \to TM$ with $\gamma(t) = (q(t), p(t))$ with $p(t) \in T^*_{q(t)}M$ is an orbit of the Hamiltonian flow that then $q : I \to M$ is a geodesic. *Hint : this is a computation in local coordinates.*

The flow defined by X_H is called the cogeodesic flow. The orbits are motions of particles experiencing no force, as the Hamiltonian only consists of the kinetic energy. The images of the orbits in M are precisely the geodesics, hence these describe the orbits of free particles on the manifold.

Exercise 2.

Let (M, ω) be a symplectic manifold. Let $\pi_1, \pi_2 : M \times M \to M$ be the projections on the first and second factor. Recall that

$$\Omega := \pi_1^*\omega - \pi_2^*\omega$$

is a symplectic form on $M \times M$. Given a map $\phi : M \to M$ we can form its graph

$$\text{graph}_\phi := \{(x, \phi(x)) \in M \times M\}.$$

The graph is an embedded $\dim M$ dimensional submanifold iff ϕ is smooth.

- Show that $\phi : M \to M$ is a symplectomorphism if and only if graph_ϕ is a Lagrangian submanifold.
- Let $p \in M$. Show that $M \times \{p\} \subseteq M \times M$ is a symplectic submanifold.

Exercise 3.

Let μ be a smooth 1-form on M thought of as a map $\mu : M \to T^*M$.

- Show that the canonical one form α is uniquely characterized by the fact that

  $$\mu^*\alpha = \mu.$$

  for all 1 forms μ on M.
- Let ω_0 be the canonical symplectic form. Give a formula for $\mu^*\omega_0$ in terms of μ.

Exercise 4.

Given μ a smooth 1-form on M, thought of as a map $M \to T^*M$, we can form its graph

$$\text{graph}_\mu := \{(q, \mu_q) | q \in M, \mu_q \in T_q^*M\}.$$

- Show that graph_μ is a Lagrangian submanifold for the canonical symplectic form ω_0 on T^*M if and only if $d\mu = 0$.