Mathematisches Institut — Universität zu Köln
Prof. Dr. Sabatini — January 7th 2015

Algebraic Topology – Homework 11

Due date : January 14th in class

<u>Exercise</u> 1.

Let G be an abelian group. Prove carefully the following statements (made in class in the lecture of the 18th of December) :

(a) The homology groups of a point p with coefficients in G are : $H_0(p;\, G) = G$ and $H_i(p;\, G) = 0$ for all $i > 0$.

(b) The reduced homology groups of an n-dimensional sphere S^n with coefficients in G are $\widetilde{H}_k(S^n;\, G) = G$ for $k = n$ and 0 otherwise, for all $n \geq 0$.

(c) Let $f\colon X \longrightarrow Y$ be a continuous map, and $f_\#\colon C_n(X) \longrightarrow C_n(Y)$ the induced maps on the groups of singular chains. Then $f_\# \otimes Id_G\colon C_n(X;\, G) \longrightarrow C_n(Y;\, G)$ is a chain map, and it induces a well-defined homomorphism $(f \otimes Id_G)_*\colon H_n(X;\, G) \longrightarrow H_n(Y;\, G)$.

(d) Let G_1 and G_2 be abelian groups, and $\Phi\colon G_1 \longrightarrow G_2$ be a group homomorphism. Then the map $Id_{C_n(X)} \otimes \Phi\colon C_n(X;\, G_1) \longrightarrow C_n(X;\, G_2)$ is a chain map, so it induces a homomorphism $(Id_{C_n(X)} \otimes \Phi)_*\colon H_n(X;\, G_1) \longrightarrow H_n(X;\, G_2)$.
Moreover the following diagram commutes :

$$\begin{array}{ccc} H_n(X;\, G_1) & \xrightarrow{(f \otimes Id_{G_1})_*} & H_n(Y;\, G_1) \\ {\scriptstyle (Id_{C_n(X)} \otimes \Phi)_*} \downarrow & & \downarrow {\scriptstyle (Id_{C_n(Y)} \otimes \Phi)_*} \\ H_n(X;\, G_2) & \xrightarrow{(f \otimes Id_{G_2})_*} & H_n(Y;\, G_2) \end{array}$$

<u>Exercise</u> 2.

Let X be a topological space endowed with the structure of a CW complex, and let G be an abelian group. Prove that the cellular homology groups with coefficients in G, denoted by $H_i^{CW}(X;\, G)$, are isomorphic to the homology groups with G coefficients $H_i(X;\, G)$ for every $i \geq 0$. (Hint : follow the proof of the analogous fact with $G = \mathbb{Z}$, justifying each step.)

Exercise 3.

Compute the homology groups with $\mathbb{Z}_2$ coefficients of all compact surfaces. (Hint : use cellular homology and the previous exercise, or the *Universal Coefficient Theorem* for homology. It would be instructive to compute at least some of these groups in both ways.)

Exercise 4.

Consider the real projective space $\mathbb{R}\mathrm{P}^2$ endowed with the usual CW structure, with one cell in dimension $0, 1$ and 2, so that the one skeleton X^1 is homeomorphic to $\mathbb{R}\mathrm{P}^1$ and the (only) two cell e^2 is attached to X^1 via a map of degree 2. Let $p\colon \mathbb{R}\mathrm{P}^2 \longrightarrow \mathbb{R}\mathrm{P}^2/X^1 \simeq S^2$ be the projection map obtained by collapsing X^1 in $\mathbb{R}\mathrm{P}^2$. Compute the following maps :

(1) $(p_\# \otimes Id_{\mathbb{Z}_2})_*\colon H_2(\mathbb{R}\mathrm{P}^2;\mathbb{Z}_2) \longrightarrow H_2(S^2;\mathbb{Z}_2)$;

(2) $p_*\colon H_i(\mathbb{R}\mathrm{P}^2;\mathbb{Z}) \longrightarrow H_i(S^2;\mathbb{Z})$ for all i.

Deduce that the map in (1) **cannot** be recovered from the maps in (2) and the *Universal Coefficient Theorem* for homology.