Mathematisches Institut
Prof. Dr. Sabatini
Assistant: Dr. Rot

Universität zu Köln
July 6th 2015

Symplectic Geometry – Homework 12

Due on July 13th 2015, in class

Exercise 1.

Let $U \subseteq \mathbb{C}^n$ be an open convex set containing 0, and $\{(A^k(U), d)\}$ the holomorphic de Rham complex defined in class. Define the operator $Q : A^k(U) \to A^{k-1}(U)$ for $k \geq 1$ as follows. For $\omega \in A^k(U)$ (the space of holomorphic k-forms) we can write $\omega = \sum_I f_I dz_I \in A^k(U)$ where $I = (i_1, \dots, i_k)$ is a multi-index. Then Q is defined by the equation

$$Q\omega = \sum_I \sum_{r=1}^{k} (-1)^{r-1} z_{i_r} \int_0^1 t^{k-1} f_I(tz) dt \, dz_{i_1} \wedge \dots \wedge \widehat{dz_{i_r}} \wedge \dots \wedge dz_{i_k}.$$

Let $\rho : U \to U$ be $\rho \equiv 0$.

(i) Show that

$$Qd\omega + dQ\omega = \omega - \rho^*\omega.$$

(ii) Conclude that the complex $\{(A^k(U), d)\}$ is acyclic.

Exercise 2.

Define $F : \mathbb{C}^n \to \mathbb{R}$ by

$$F(z) = \log(1 + |z|^2).$$

and consider the matrix

$$h_{ij} := \frac{\partial}{\partial z_i} \frac{\partial}{\partial \overline{z_j}} F(z) = \frac{\delta_{ij}}{1 + |z|^2} - \frac{\overline{z_i} z_j}{(1 + |z|^2)^2}.$$

(a) Prove that

$$\det(h_{ij}) = (1 + |z|^2)^{-(n+1)}.$$

(*Point (a) is hard, but you are clever! The shortest proof will award +5 points, so you may end up with 55/50...*)

(b) Prove that F is strictly plurisubharmonic.

(c) Show that the Fubini-Study form on $\mathbb{C}P^n$ is Kähler-Einstein with $\lambda = -(n + 1)$, i.e.

$$\eta = -(n + 1)\omega_{\text{FS}}.$$