Mathematisches Institut | Universität zu Köln
Prof. Dr. Sabatini | November 5th 2014

Algebraic Topology – Homework 4

Due date : November 12th in class

In the following exercises, since the spaces you are going to work with are path connected, the base point for computing the fundamental group is omitted.

Exercise 1.

Use Van Kampen's Theorem to prove the following facts.

(a) Let M be an n-dimensional (path)connected manifold, with $n \geq 3$. Then $\pi_1(M) \simeq \pi_1(M \setminus \{p\})$, where $p \in M$. Is it true for $n = 2$? Find a counterexample if it is not.

(b) Let X be a connected CW complex, and Y a connected CW complex obtained from X by attaching cells D^n_α of dimension $n \geq 3$. Prove that $\pi_1(X) \simeq \pi_1(Y)$.

(c) In analogy with the connected sum defined on surfaces, given two n-dimensional manifolds M and N, their connected sum $M \# N$ is defined as follows : Consider two open balls B^n_1 and B^n_2 of dimension n respectively in M and N. Let $h \colon \partial B^n_1 \longrightarrow \partial B^n_2$ be a homeomorphism. Then $M \# N$ is defined to be $((M \setminus B^n_1) \amalg (N \setminus B^n_2))/\sim$, where we identify the points in ∂B^n_1 and ∂B^n_2 by using h, i.e. $x \sim h(x)$ for every $x \in \partial B^n_1$. Prove that if $n \geq 3$ and M and N are connected, then $\pi_1(M \# N) \simeq \pi_1(M) * \pi_1(N)$.

Exercise 2.

Compute the fundamental group of the following spaces :

(1) The real projective space $\mathbb{R}P^n$, for every $n \in \mathbb{Z}_{\geq 1}$.
(2) The closed 2-disc $D^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ minus m distinct points, where $m \in \mathbb{Z}_{\geq 1}$.
(3) $\mathbb{R}^3 \setminus X$, where X is the union of n distinct lines through the origin.
(4) The 1-dimensional CW complex that is the union of edges and vertices on a tetrahedron.

Exercise 3.

The complex projective space $\mathbb{C}P^n$ is defined as the quotient space of $\mathbb{C}^{n+1} \setminus \{0,\dots,0\}$, wherein a point $(z_0,\dots,z_n) \in \mathbb{C}^{n+1} \setminus \{0,\dots,0\}$ is identified with all points $(\lambda z_0,\dots,\lambda z_n)$, for $\lambda \in \mathbb{C} \setminus \{0\}$. Denote the projection map by $q\colon \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{C}P^n$, endow $\mathbb{C}P^n$ with the quotient topology with respect to q, and denote by $[z_0,\dots,z_n]$ the point $q(z_0,\dots,z_n)$. The coordinates $[z_0,\dots,z_n]$ are called *homogeneous coordinates*, and are defined only up to a constant $\lambda \in \mathbb{C} \setminus \{0\}$.

(i) Show that any point in $\mathbb{C}P^n$ has homogeneous coordinates such that $\sum_{i=0}^{n} \|z_i\|^2 = 1$, i.e. $(z_0,\dots,z_n) \in S^{2n+1} \subset \mathbb{R}^{2(n+1)} = \mathbb{C}^{n+1}$. Thus the map $q\colon \mathbb{C}^{n+1} \setminus \{0,\dots,0\} \longrightarrow \mathbb{C}P^n$ factors

$$\mathbb{C}^{n+1} \setminus \{0,\dots,0\} \xrightarrow{p} S^{2n+1} \xrightarrow{\pi} \mathbb{C}P^n.$$

Conclude that $\mathbb{C}P^n$ is compact.

(ii) For each $i \in \{0,\dots,n\}$, let $U_i = \{[z_0,\dots,z_n] \in \mathbb{C}P^n \mid z_i \neq 0\}$. Show that U_i is open for every i, and is homeomorphic to $\mathbb{C}^n$; conclude that $\mathbb{C}P^n$ is a $2n$-dimensional manifold (skip the proof of the Hausdorff condition). Show that $\mathbb{C}P^n \setminus U_i$ is homeomorphic to $\mathbb{C}P^{n-1}$.

(iii) By what you proved above, $\mathbb{C}P^1$ is a compact surface. Find an explicit homeomorphism between $\mathbb{C}P^1$ and one of the surfaces appearing in the classification theorem of compact surfaces.

(iv) Prove that $\mathbb{C}P^n$ has the structure of a CW complex, specifying what the "attaching maps" φ_α are (see the notation introduced in class), and compute $\pi_1(\mathbb{C}P^n)$ for every $n \geq 1$.