Mathematisches Institut
Prof. Dr. Sabatini
Assistant: Dr. Rot

Universität zu Köln
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Symplectic Geometry – Homework 4

Due on May 11th 2015, in class

Exercise 1.

See the sphere S^2 as a subset of $\mathbb{R}^3$. Denote the standard inner product on $\mathbb{R}^3$ by $\langle \cdot, \cdot \rangle$ and the exterior product by $\cdot \times \cdot$. Show that the formula

$$\omega_x(v, w) = \langle x, v \times w \rangle \qquad \text{for} \qquad x \in S^2 \qquad \text{and} \qquad u, v \in T_x S^2$$

defines a symplectic form on S^2.

Exercise 2.

Which closed (i.e. compact, boundary-less) surfaces admit symplectic structures ?

Exercise 3.

Let ω_0 be the standard symplectic structure on $\mathbb{R}^{2n}$. For which functions $f : \mathbb{R}^{2n} \to \mathbb{R}$ is $(\mathbb{R}^{2n}, f\omega_0)$ a symplectic manifold ?

Exercise 4.

Let (M, ω) be a $2n$-dimensional symplectic manifold.

- Show that if M is closed, that the class $[\omega^n] \in H^{2n}(M)$ is non-zero.
- Show that $H^{2i}(M) \neq 0$ for all $1 \leq i \leq n$.
- Which spheres S^{2n} admit symplectic structures ?

Remark : If you did the second part of this exercise correctly you have seen actually a bit more than was asked to prove : The *cuplength* of a symplectic manifold is always bounded from below by $\frac{1}{2}\dim(M)$.

Exercise 5.

It is possible to define the differential d acting on forms coordinate independently. We do this here for 0-forms (i.e. functions) and 1-forms.

- Let $f : M \to \mathbb{R}$ be a function. Show that df is the unique form such that
$$df(X) = X(f),$$
for all vectorfields X on M.
- Let $\alpha \in \Omega^1(M)$. Show that $d\alpha$ is the unique 2-form such that
$$d\alpha(X, Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X, Y]),$$
for all vector fields X and Y on M. (You will need to show that the formula on the right hand side actually defines a 2-form)