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Symplectic Geometry – Homework 5

Due on May 18th 2015, in class

In these exercises we will construct the Kodaira-Thurston manifold (See for example the paper of Thurston (1976) : "On some examples of symplectic manifolds"). The exercises seem long but most parts are routine verification and should not be difficult.

Exercise 1.

For $j = (j_1, j_2) \in \mathbb{Z}^2$, let $A_j = \begin{pmatrix} 1 & j_2 \\ 0 & 1 \end{pmatrix}$. Consider the following operation on $\mathbb{Z}^2 \times \mathbb{Z}^2$:

$$(j', k') \circ (j, k) = (j + j', A_{j'}k + k')$$

- It is not hard to show that $\Gamma = (\mathbb{Z}^2 \times \mathbb{Z}^2, \circ)$ is a group. You do not have to show this, but what is the identity element ? What is the inverse of a general element (j, k) ?
- Recall that the commutator subgroup $[\Gamma, \Gamma]$ is the subgroup generated by elements $aba^{-1}b^{-1}$. Show that

$$[\Gamma, \Gamma] = \{0\} \times \{0\} \times \mathbb{Z} \times \{0\} \cong \mathbb{Z}.$$

- Show that the map $\Gamma \to \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, defined by $(j, k) \longmapsto (j_1, j_2, k_2)$ is a surjective group homomorphism.
- Conclude that $\Gamma/[\Gamma, \Gamma] \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

Exercise 2.

Set $X = \mathbb{R}^2 \times \mathbb{R}^2$. For $(j, k) \in \Gamma$, define $\rho_{jk} : X \to X$ by

$$\rho_{jk}(x, y) = (x + j, A_j y + k).$$

- Show that this defines an action of Γ on X.
- Show that the action is *properly discontinuous.*

Thus $M = X/\Gamma$ is a smooth 4 dimensional manifold. It is actually symplectic as we show next.

- Let ω be the symplectic form

$$\omega = dx_1 \wedge dx_2 + dy_1 \wedge dy_2,$$

on X. Show that the maps ρ_{jk} are symplectomorphisms. Define a symplectic form on M.

Due to lack of time we have not discussed much covering space theory in the algebraic topology class. One result one can obtain is the following

Proposition 1 (Hatcher 1.40)**.** *Let Y be a path-connected and locally path-connected space. If a group G acts properly discontinuously on Y, then*

$$\pi_1(Y/G)/p_*(\pi_1(Y)) \cong G.$$

Here $p : Y \to Y/G$ is the projection.

- Use this proposition to compute $\pi_1(M)$.
- Compute $H_1(M;\mathbb{Z})$.

Thus M is a symplectic manifold with $H_1(M;\mathbb{Z})$ odd dimensional. It can be shown that a Kähler manifold, i.e. a manifold with a complex, Riemannian and symplectic structure interacting nicely, must have all odd Betti numbers even. Thus M is symplectic but not Kähler. It took a while to find examples of such manifolds. Papers even appeared claiming that such examples do not exist.