Mathematisches Institut — Universität zu Köln
Prof. Dr. Sabatini — January 28th 2015

Algebraic Topology – Homework 14 - the last !

Due date : **Friday February 6th** at noon
in Mrs. Angela Georg's office "Geschaeftszimmer"
in Prof. Sabatini's mail box.
Graded by **Monday February 9th**.
Make an appointment with Dr. Rot to have the homework back on Monday 9th.

Exercise 1.

Prove or disprove :

(a) $S^1 \times S^1$ is homotopically equivalent to $S^1 \vee S^1 \vee S^2$;
(b) $\mathbb{C}\mathrm{P}^2$ is homotopically equivalent to $S^2 \vee S^4$.

Exercise 2.

(i) Compute the (cellular) homology and cohomology groups of $\mathbb{R}\mathrm{P}^\infty$ with $\mathbb{Z}$ coefficients.

Let D^k be the (closed) k-dimensional disk (or k-dimensional ball). A topological space X is called a *rational homology ball* if $H_i(X;\mathbb{Q}) \simeq H_i(D^k;\mathbb{Q})$ for all $i \geq 0$.
(ii) For which $1 \leq m \leq \infty$ is $\mathbb{R}\mathrm{P}^m$ a rational homology disk ?

Let S^k be the k-dimensional sphere. A topological space X is called a *rational homology k-sphere* if $H_i(X;\mathbb{Q}) \simeq H_i(S^k;\mathbb{Q})$ for all $i \geq 0$.
(iii) For which $1 \leq m \leq \infty$ is $\mathbb{R}\mathrm{P}^m$ a rational homology k-sphere (for some k) ?

Exercise 3.

Let $C = \{(C_n, \partial)\}$ be a chain complex, where C_n is a free abelian group for every n. Prove that, if $H_i(C)$ is finitely generated for every i, then

$$H_i(C;\mathbb{Z}_m) \simeq H^i(C;\mathbb{Z}_m)$$

for every $m \in \mathbb{Z}_{\geq 1}$.

<u>Exercise</u> 4.

Generalise the argument shown in class to compute explicitly the following (simplicial) cohomology rings :

(a) $H^*(\Sigma_g; \mathbb{Z})$, where Σ_g denotes the orientable surface of genus g, for $g \geq 2$ (the case $g = 1$ was done in class).

(b) $H^*(\widetilde{\Sigma}_g; \mathbb{Z}_2)$, where $\widetilde{\Sigma}_g$ denotes the non-orientable surface of genus g, for $g \geq 2$ (the case $g = 1$ was done in class).

Use these computations to show that :

(c) If $f\colon K \longrightarrow T$ is a continuous map, then $f^*\colon H^2(T; \mathbb{Z}_2) \longrightarrow H^2(K; \mathbb{Z}_2)$ is trivial, where K denotes the Klein bottle ;

(d) If $f\colon S^2 \longrightarrow K$ is a continuous map, then $f^*\colon H^2(K; \mathbb{Z}_2) \longrightarrow H^2(S^2; \mathbb{Z}_2)$ is trivial ;

(e) If $f\colon T \longrightarrow K$ is a continuous map, then $f^*\colon H^2(K; \mathbb{Z}_2) \longrightarrow H^2(T; \mathbb{Z}_2)$ is trivial ;

<u>Exercise</u> 5.

Let $p = [x_0 : x_1 : \ldots : x_n]$ be a point in $\mathbb{R}\mathrm{P}^n$, and let $\mathbb{R}\mathrm{P}^m = \{p \in \mathbb{R}\mathrm{P}^n \mid x_{m+1} = \ldots = x_n = 0\} \subset \mathbb{R}\mathrm{P}^n$, for some $1 \leq m < n$. Does there exist a retraction $r\colon \mathbb{R}\mathrm{P}^n \longrightarrow \mathbb{R}\mathrm{P}^m$?