Mathematisches Institut | Universität zu Köln
Prof. Dr. Sabatini | 08.05.2019

Algebraic Topology – Homework 6

Due date : May 15th in class.

Exercise 19. (5 Points)

Prove that given $f : S^n \longrightarrow S^n$ with $\deg(f) \neq \pm 1$, there exists $x \in S^n$, such that $f(x) = \pm x$.

Exercise 20. (5 Points)

Given a map $f : S^{2n} \longrightarrow S^{2n}$, show that there is some $x \in S^{2n}$ with either $f(x) = x$ or $f(x) = -x$. Deduce that every map $\mathbb{RP}^n \longrightarrow \mathbb{RP}^n$ has a fixed point. Construct maps $\mathbb{RP}^{2n-1} \longrightarrow \mathbb{RP}^{2n-1}$ without fixed points fron linear transformations $\mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ without eigenvectors.

Exercise 21. (5 Points)

Let $f : S^n \longrightarrow S^n$ be a homeomorphism. Prove that $\deg f = \pm 1$.

Exercise 22. (5+5+5 Points)

Let $A \in O(n+1)$ be an orthonormal matrix. Observe that A induces a map $f_A : S^n \longrightarrow S^n$ given by $f_A(\bar{x}) = A \cdot \bar{x}$ where

$$\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_{n+1} \end{pmatrix} \in S^n .$$

(i) Prove that $O(n+1)$ has two path-connected components for every $n \geq 0$.
Hint : Use the fact that any orthogonal matrix is orthogonally similar to a block diagonal matrix of the form $A_1 \oplus A_2 \oplus \ldots \oplus A_r$, where each A_i is either $[\pm 1]$ or a 2×2 rotation matrix of the type

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

for some $\theta \in \mathbb{R}$.

(ii) Prove that given $A, B \in O(n+1)$ in the same connected component, f_A and f_B are homotopic.

(iii) Conclude that $\deg f_A = \det A$ for every $A \in O(n+1)$.

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Consider $H_{n-1}(\mathbb{R}^n - \{p\})$ and choose a generator a of this group to identify canonically $H_{n-1}(\mathbb{R}^n - \{p\})$ with $\mathbb{Z}$ by sending a to 1.
For an open set $U \subset \mathbb{R}^n$ and $p \in U$, we use the following two isomorphisms to pull back a to a generator of $H_n(U, U - \{p\})$, namely :

$$\iota_* : H_n(U, U - \{p\}) \longrightarrow H_n(\mathbb{R}^n, \mathbb{R}^n - \{p\}),$$

induced by the inclusion map $\iota : (U, U - \{p\}) \longrightarrow (\mathbb{R}^n, \mathbb{R}^n - \{p\})$, and the inverse of the boundary operator in the long exact sequence

$$0 \cong H_n(\mathbb{R}^n) \longrightarrow H_n(\mathbb{R}, \mathbb{R}^n - \{p\}) \longrightarrow H_{n-1}(\mathbb{R}^n - \{p\}) \longrightarrow H_{n-1}(\mathbb{R}^{n-1}) \cong 0\,.$$

Let $U, V \subset \mathbb{R}^n$, be two open sets, p a point in U and $f \colon U \longrightarrow V$ a continuous function. Following the above procedure, let a' and b' be the generators of the groups $H_n(U, U - \{p\}), H_n(V, V - \{f(p)\})$, which we use to identify $H_n(U, U - \{p\})$ and $H_n(V, V - \{f(p)\})$ with $\mathbb{Z}$. With these identifications the map

$$f_* : H_n(U, U - \{p\}) \longrightarrow H_n(V, V - \{f(p)\})$$

becomes multiplication by an integer, called the *degree of f at p* and denoted by $\deg f_p$.

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Exercise 23. (5+5 Points)

In this Exercise you may assume the following Proposition :

Proposition. *Let $U, V \subset \mathbb{R}^n$ be open sets and $f : U \longrightarrow V$ be a C^∞-function such that for $p \in U$, the Jacobian matrix $J_f(p)$ is invertible. Then the local degree of f at p equals the sign of the determinant of the Jacobian matrix $J_f(p)$.*

(i) Show that a polynomial $f(z)$ with complex coefficients, viewed as a map $\mathbb{C} \longrightarrow \mathbb{C}$, can be extended to a continuous map of one-point compactifications $\tilde{f} : \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$.

(ii) Show that the degree of $\tilde{f}$ equals the degree of f as a polynomial.

<u>Exercise</u> 24. (5+5 Points)

Let $X = \bigcup_{n\geq 0} X^n$ be a CW-complex, where its n-skeleton X^n is defined inductively from a collection of points X^0 endowed with discrete topology, and by attaching n-cells via maps $\varphi_\alpha : \partial D^n_\alpha \longrightarrow X^{n-1}$. This means that X^n is the quotient space of $X^{n-1} \sqcup \bigsqcup_\alpha D^n_\alpha$ under the identifications $x \sim \varphi_\alpha(x)$ for $x \in \partial D^n_\alpha$, endowed with quotient topology. Let q^n be the quotient map

$$q^n : X^{n-1} \sqcup \bigsqcup_\alpha D^n_\alpha \longrightarrow X^n .$$

(i) Prove that

$$q^n|_{\mathring{D}^n_\alpha} : \mathring{D}^n_\alpha \longrightarrow X^n$$

is a homeomorphism onto its image, where the topology on $q^n(\mathring{D}^n_\alpha)$ is the subset topology inherited as a subset of X^n.

(ii) Prove that

$$q^n|_{\mathring{D}^n_\alpha} : \mathring{D}^n_\alpha \longrightarrow X$$

is a homeomorphism onto its image, where the topology on $q^n(\mathring{D}^n_\alpha)$ is the subset topology inherited as a subset of X. (This is harder than (i)!)