Algebraic Topology – Homework 9

Due date : June 5th in class.

Exercise 33. (10 Points)

Compute the homology of $S^2 \times S^1$ using the Mayer-Vietoris sequence associated to the sets $A = S^2 \times (S^1 \setminus \{p\})$ and $B = S^2 \times (S^1 \setminus \{q\})$, where p and q are distinct points in S^1.

Exercise 34. (15 Points)

Let $\triangle^n = [v_0, \ldots, v_n]$ have its natural $\triangle$-complex structure with k-simplices $[v_{i_0}, \ldots, v_{i_k}]$ for $i_1 < \ldots < i_k$. Compute the ranks of the simplicial (or cellular) chain groups $\triangle_i(\triangle^n)$ and the subgroups of cycles and boundaries. *Hint :* Pascal's triangle.
Apply this to show that the k-skeleton of $\triangle^n$, denoted by $(\triangle^n)^k$, has homology groups $\widetilde{H}_i((\triangle^n)^k)$ equal to 0 for $i < k$, and free of rank $\binom{n}{k+1}$ for $i = k$.

Exercise 35. (8+5+12 Points)

Let G be an abelian group.

(i) Let (X, A) be a good pair. Show that the quotient map $q : (X, A) \longrightarrow (X/A, A/A)$ induces isomorphisms

$$q_* : H_n(X, A; G) \longrightarrow H_n(X/A, A/A; G) \cong \widetilde{H}_n(X/A; G)$$

for all n.
Note : In this Exercise you can assume that the Excision Theorem holds for homology with coefficients.

(ii) Let S^n denote the n-dimensional sphere. Show that

$$\widetilde{H}_k(S^n; G) = \begin{cases} G & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}$$

(iii) Let X be a finite CW-complex. Show that :

a) $H_k(X^n, X^{n-1}; G)$ is zero for $k \neq n$ and equals G^{c_n} for $k = n$, where c_n denotes the number of n-cells of X.

b) $H_k(X^n; G) = \{0\}$ for $k > n$.

c) The inclusion $i : X^n \hookrightarrow X$ induces and isomorphism

$$i_* : H_k(X^n; G) \longrightarrow H_k(X; G)$$

if $k < n$.

d) $H_n^{CW}(X; G) \cong H_n(X; G)$.