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Symplectic Geometry – Homework 6

Due on June 1st 2015, in class

Exercise 1.

Consider cylindrical coordinates (θ, x_3) on $M = S^2 \setminus \{(0,0,\pm 1)\}$. Show that the area form induced by the Euclidean metric obtained from the embedding $M \subset \mathbb{R}^3$ is the form $\omega = d\theta \wedge dx_3$. Show that the same holds for the cylinder $C = S^1 \times [-1,1] \subset \mathbb{R}^3$. Thus the horizontal projection $\phi : M \longrightarrow C$ defined by

$$\phi(x_1, x_2, x_3) = \left(\frac{x_1}{x_1^2 + x_2^2}, \frac{x_2}{x_1^2 + x_2^2}, x_3 \right)$$

is area preserving (thus symplectic).

Exercise 2.

Extend the symplectic form ω of Exercise 1 to S^2. Compute the Hamiltonian vector field of the Hamiltonian $H : S^2 \subset \mathbb{R}^3 \to \mathbb{R}$

$$H(x_1, x_2, x_3) = x_3.$$

Describe the dynamics of the flow generated by the Hamiltonian vector field.

Exercise 3.

Let A_t be a smooth path in $Sp(2n, \mathbb{R})$ with $A_0 = \mathbb{1}$, the identity matrix.

- By differentiating the defining equation

$$A_t^T J_0 A_t = J_0$$

  with respect to t, show that each $X \in T_{\mathbb{1}} Sp(2n, \mathbb{R}) \subset T_{\mathbb{1}} GL(2n, \mathbb{R}) \cong M_{2n \times 2n}(\mathbb{R})$ satisfies

$$X^T J_0 + J_0 X = 0\,. \qquad (1)$$

Indeed it can be proved that any matrix $X \in M_{2n \times 2n}(\mathbb{R})$ that satisfies (1) is in $T_{\mathbb{1}} Sp(2n, \mathbb{R})$.

- Compute the dimension of $T_{\mathbb{1}} Sp(2n, \mathbb{R})$.

Exercise 4.

Let M be a manifold, and $\alpha \in \Gamma(T^*M \otimes T^*M)$, i.e. α is a section of $T^*M \otimes T^*M$. Suppose α is non-degenerate, thus for every $p \in M$ the map $\sigma_p : T_pM \to T_p^*M$ defined by $\sigma_p(v) = \alpha(v, \cdot)$ is an isomorphism. Nondegeracy ensures that each $f \in C^\infty(M)$ defines a unique vector field Y_f by $\alpha(Y_f, \cdot) = df$.
Define a bracket $\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ by

$$\{f, g\} := \alpha(X_f, X_g).$$

- Show that the bracket is antisymmetric if and only if α is a 2-form.
- Suppose that α is a 2-form. Show that

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = d\alpha(X_f, X_g, X_h)$$

Hint : the differential of a 2-form α is characterized by

$$\begin{aligned} d\alpha(X, Y, Z) =& X \cdot \alpha(Y, Z) - Y \cdot \alpha(X, Z) + Z \cdot \alpha(X, Y) \\ & - \alpha([X, Y], Z) + \alpha([X, Z], Y) - \alpha([Y, Z], X). \end{aligned}$$

Conclude that $\{\cdot, \cdot\}$ is a Lie bracket if and only if α is symplectic. Such a Lie bracket is called the *Poisson bracket* induced by the symplectic structure.