Mathematisches Institut | Universität zu Köln
Prof. Dr. Sabatini | October 15th 2014

Algebraic Topology – Homework 1

Due date : October 22nd in class

<u>Exercise</u> 1.

Let S_1 and S_2 be two compact surfaces. Prove that $S_1 \# S_2$ is orientable if and only if both surfaces are.

Since the definition of orientable surface given in class was not formal, the proof should not be formal either. You should basically find an argument that convinces yourself that the above statement is true. A careful proof can be carried on by using the formal definition of orientability, which will be given later in the course.

<u>Exercise</u> 2.

The proof of the following homeomorphisms should consist of cutting and pasting of planar diagrams, along with suitable explanations.

- Recall that Σ_g, the *orientable surface of genus* g, is obtained as the quotient space of the $4n$-polygon with identifications :

  $$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \ .$$

  Here $g \in \mathbb{Z}$ and $g \geq 1$. (Also, see Hatcher's book on page 5.)
  Prove that Σ_g is homeomorphic to $\Sigma_1 \# \cdots \# \Sigma_1$.
  (Hint : start with the case $g = 2$: what happens if you cut the 8-gon in half, separating the letters a_1, b_1 from a_2, b_2 ?)

- Recall that K, the Klein bottle, is obtained as the quotient space of a square with identifications :

  $$a \, b \, a^{-1} \, b \ .$$

  Prove that K is homeomorphic to $\mathbb{RP}^2 \# \mathbb{RP}^2$.
  (Hint : Which surface is obtained by removing a disc D^2 from $\mathbb{RP}^2$?)

Exercise 3.

In this exercise, *assume* the independence of the Euler characteristic from the triangulation of the surface.

(a) Let S_1 and S_2 be compact surfaces, and S^2 the 2-dimensional sphere. Let τ_1 and τ_2 be triangulations on S_1 and S_2 respectively. Choose a suitable triangulation on $S_1 \# S_2$ to prove that

$$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - \chi(S^2). \tag{1}$$

(b) Let Σ_g be the orientable surface of genus g (Recall that for $g = 0$ this is defined to be the sphere S^2). Prove that for every $g \in \mathbb{Z}_{\geq 0}$

$$\chi(\Sigma_g) = 2 - 2g\,.$$

(c) Let $\widetilde{\Sigma}_g$ be the connected sum of g projective planes, for $g \in \mathbb{Z},\ g \geq 1$. Prove that

$$\chi(\widetilde{\Sigma}_g) = 2 - g\,.$$

(Hint : Prove first that

$$\chi(\mathbb{R}\mathrm{P}^2) = 1$$

by viewing $\mathbb{R}\mathrm{P}^2$ as a quotient space of a 2-gon, and finding a triangulation on the 2-gon that passes to a triangulation of the quotient.)

Exercise 4.

In this exercise, *assume* the independence of the Euler characteristic from the triangulation of the surface, and the classification theorem of compact surfaces stated in class. Using the previous exercises, prove the following alternative formulation of the classification theorem :

Theorem 1. *Two compact surfaces are homeomorphic if and only if their Euler characteristic coincides and they are both orientable, or both non-orientable.*