Mathematisches Institut | Universität zu Köln
Prof. Dr. Sabatini | October 22nd 2014

Algebraic Topology – Homework 2

Due date : October 29th in class
Mandatory exercises : 1, 2, 3 and 4. Optional : 5 and 6

Exercise 1.

Let Y be a topological space. Prove that a continuous map $f\colon S^1 \longrightarrow Y$ is homotopic to a constant map if and only if f extends to a continuous map $g\colon D^2 \longrightarrow Y$, where D^2 is the closed unit disc $\{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}$ (so ∂D^2, the boundary of D^2, is precisely S^1) and g satisfies $g_{|_{\partial D^2}} = f$.

Exercise 2.

Let X be a topological space, and $f, g\colon X \longrightarrow S^n \subset \mathbb{R}^{n+1}$ be continuous maps satisfying $f(x) \neq -g(x)$ for all $x \in X$. (Here S^n denotes the unit sphere.)
Prove that f is homotopic to g. Deduce that if $f\colon X \longrightarrow S^n$ is not surjective, then it is nullhomotopic.

Exercise 3.

Find an explicit, non-trivial deformation retraction (and prove that it is a deformation retraction) of the following spaces :
(a) The two dimensional torus T minus a point ;
(b) The n-dimensional (real) projective space $\mathbb{RP}^n$ minus a point.

Exercise 4.

Let X and $A \subsetneq X$ be topological spaces (A is endowed with the subspace topology). We say that A is a *weak deformation retract*[1] of X if A is a retract of X (we denote the retraction by $r\colon X \longrightarrow A$) and $i \circ r\colon X \longrightarrow X$ is homotopic to the identity map Id_X (not relative to A!), where $i\colon A \longrightarrow X$ denotes the inclusion.

- Let $X \subset \mathbb{R}^2$ be the union of the segments $X_0 = \{0\} \times [0,1]$, $X_n = \{\frac{1}{n}\} \times [0,1]$ for every $n \in \mathbb{N} \setminus \{0\}$ and $Y = [0,1] \times \{0\}$. (Thus X looks like a *comb*.) Prove that X_0 is a weak deformation retract of X, but not a deformation retract (i.e. the homotopy from the identity on X to the retraction *cannot* leave the points of X_0 fixed).

1. This notion may be defined differently by different authors.

Exercise 5.

(1) Prove that a contractible space is path-connected;
(2) Let $x \neq y$ be two points of a topological space X. Prove that the "constant maps" $c_x\colon X \longrightarrow X$, where $c_x(X) = \{x\}$ and $c_y\colon X \longrightarrow X$, where $c_y(X) = \{y\}$, are homotopic if and only if x and y belong to the same path-connected component.

Exercise 6.

Let X be a topological space. Show that the following conditions are equivalent :
(a) X is contractible;
(b) The identity map on X is homotopic to a constant map (such maps are called *nullhomotopic*);
(c) Every map $f\colon X \longrightarrow Y$, for arbitrary Y, is nullhomotopic;
(d) Every map $g\colon Y \longrightarrow X$, for arbitrary X, is nullhomotopic.