Mathematisches Institut | Universität zu Köln
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## Algebraic Topology – Homework 5

Due date : May 8th in class.

### Exercise 16. (4+4+4+4+4+6 Points)

We recall that every finitely generated abelian group $G$ can be written as

$$G \cong \mathbb{Z}^p \oplus \operatorname{Tor} G$$

where $\operatorname{Tor} G$ is a finite abelian group called the *torsion part* of $G$ and the integer $p$ is called the *rank of* $G$ and it is denoted by $\operatorname{rank}(G)$.
Let $A, B$ and $C$ be finitely generated abelian groups such that the following sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

is exact. In this series of exercises we want to show that

$$\operatorname{rank} B = \operatorname{rank} A + \operatorname{rank} C :$$

(a) Write $C$ as $C = \mathbb{Z}^p \oplus \operatorname{Tor} C$, where $\operatorname{Tor} C$ is the torsion part of $C$. Show that there exists $s : \mathbb{Z}^p \longrightarrow B$ such that

$$\beta \circ s = \mathrm{id}_{\mathbb{Z}^p} \ .$$

(b) We define $\alpha' : A \oplus \mathbb{Z}^p \longrightarrow B$ by $\alpha'(\alpha, x) = \alpha(a) + s(x)$ and $\beta' : B \longrightarrow \operatorname{Tor} C$ by $\beta(b) = \pi \circ \beta(b)$ where $\pi$ is the projection map $\pi : C \longrightarrow C/\mathbb{Z}^p = \operatorname{Tor} C$ . Show that the sequence

$$0 \longrightarrow A \oplus \mathbb{Z}^p \xrightarrow{\alpha'} B \xrightarrow{\beta'} \operatorname{Tor} C \longrightarrow 0$$

is exact.

(c) Show that there exist well defined maps

$$\alpha^* : \frac{A \oplus \mathbb{Z}^p}{\operatorname{Tor} A} \longrightarrow \frac{B}{\operatorname{Tor} B} \ , \ \beta^* : \frac{B}{\operatorname{Tor} B} \longrightarrow \frac{\operatorname{Tor} C}{\beta'(\operatorname{Tor} B)}$$

such that they give rise to a commutative diagram

$$\begin{array}{ccccc}
A \oplus \mathbb{Z}^p & \xrightarrow{\alpha'} & B & \xrightarrow{\beta'} & \operatorname{Tor} C \\
\downarrow & & \downarrow & & \downarrow \\
\dfrac{A \oplus \mathbb{Z}^p}{\operatorname{Tor} A} & \xrightarrow[\alpha^*]{} & \dfrac{B}{\operatorname{Tor} B} & \xrightarrow[\beta*]{} & \dfrac{\operatorname{Tor} C}{\beta'(\operatorname{Tor} B)}
\end{array}$$

and such that the sequence

$$0 \longrightarrow \frac{A \oplus \mathbb{Z}^p}{\operatorname{Tor} A} \xrightarrow{\alpha^*} \frac{B}{\operatorname{Tor} B} \xrightarrow{\beta*} \frac{\operatorname{Tor} C}{\beta'(\operatorname{Tor} B)} \longrightarrow 0$$

is exact.

(d) Let $D$ be a finite abelian group and $r, s$ be nonnegative integers such that we have a short exact sequence of abelian groups

$$0 \longrightarrow \mathbb{Z}^r \longrightarrow \mathbb{Z}^s \longrightarrow D \longrightarrow 0.$$

Show that $r = s$. *Hint :* Show that the sequence gives rise to an isomorphism of vector spaces $\mathbb{Q}^r \longrightarrow \mathbb{Q}^s$.

(e) Write $A$ as $A = \mathbb{Z}^m \oplus \operatorname{Tor} A$ and $B$ as $B = \mathbb{Z}^n \oplus \operatorname{Tor} B$. Use the previous exercises to conclude that

$$\operatorname{rank} B = n = m + p = \operatorname{rank} A + \operatorname{rank} C.$$

(f) Let $C_i$ be finitely generated abelian groups. Prove that given a chain complex :

$$0 \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

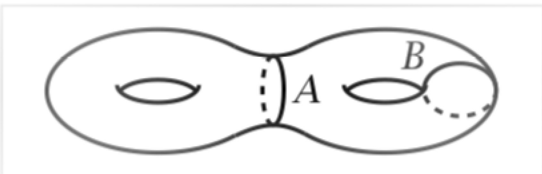
with homology groups $H_i(C)$, then

$$\sum_{i=0}^{n} (-1)^i \operatorname{rank}(C_i) = \sum_{i=0}^{n} (-1)^i \operatorname{rank}(H_i(C)) .$$

**Exercise 17. (6+6 Points)**

(a) Compute the homology groups $H_n(X, A)$ where $X$ is $S^2$ or $S^1 \times S^1$ and $A$ is a finite set of points.

(b) Compute the groups $H_1(X, A)$ and $H_1(X, B)$ for $X$ a closed orientable surface of genus two with $A$ and $B$ the circles shown in the picture below.



**Exercise 18. (4+4+4 Points)**

Let $f : (X, A) \longrightarrow (Y, B)$ be a map such that both $f : X \longrightarrow Y$ and the restriction $f : A \longrightarrow B$ are homotopy equivalences. This is called a homotopy equivalence of pairs $f\colon (X, A) \longrightarrow (Y, B)$

(a) Show that $f_* : H_n(X, A) \longrightarrow H_n(Y, B)$ is an isomorphism for all $n$.

(b) Show that a homotopy equivalence of pairs $(X, A) \longrightarrow (Y, B)$ is also a homotopy equivalence for the pairs obtained by replacing $A$ and $B$ by their closures.

(c) Show that the inclusion $f : (D^n, S^{n-1}) \hookrightarrow (D^n, D^n \setminus \{0\})$ is not a homotopy equivalence of pairs.