Examples

8.3 We start by considering the following simple system of ordinary differential equations in the (x, y) plane:

$$\begin{cases} \dfrac{dx}{dt} = \cos^2 x, \\[2ex] \dfrac{dy}{dt} = \sin x. \end{cases}$$

It is easily seen that the integral curves are the curves

$$y = \sec x + C$$

for various values of the constant of integration C, and the vertical lines

$$x = (n + \tfrac{1}{2})\pi$$

for all integers n. We can consider this system of differential equations the equations of motion of a particle in the plane; t represents the time, and (x, y) are the coordinates of the particle at time t. The particle must move along one of the integral curves. Which curve it will move along depends on its initial position.

Using this differential equation, we shall define an operation of the additive group of real numbers, $\mathbf{R}$, on the Euclidean plane. For any real number t and any point (x, y) of the plane, we define $t \cdot (x, y)$ to be the position at time t of a particle which was at the point (x, y) at time 0. It is clear that

$$s \cdot [t \cdot (x, y)] = (s + t) \cdot (x, y),$$
$$0 \cdot (x, y) = (x, y).$$

Also, the map $\mathbf{R} \times \mathbf{R}^2 \rightarrow \mathbf{R}^2$ defined by $(t, (x, y)) \rightarrow t \cdot (x, y)$ is continuous (it is even differentiable). This is a consequence of standard theorems on differential equations. It is also clear that this action of $\mathbf{R}$ on the plane is fixed point free.

We now consider the action of the subgroup $\mathbf{Z}$ of $\mathbf{R}$ on the plane; this will give our desired example.

We shall first prove that the action of $\mathbf{Z}$ on $\mathbf{R}^2$ is properly discontinuous. Given any point $P = (x, y)$, let C be the unique integral curve passing through P. Let C_1 and C_2 be two integral curves near by, one on each side of C. Let T_0 be a smooth curve through P which is orthogonal to all the integral curves between C_1 and C_2. For any real number t, let $T_t = t \cdot T_0$. Let U be the neighborhood of P bounded by $T_{-1/3}$, $T_{+1/3}$, and the curves C_1 and C_2. Then, it is readily seen that the successive "translates" of U,

$$\{n \cdot U : n \in \mathbf{Z}\}$$

are pairwise disjoint.

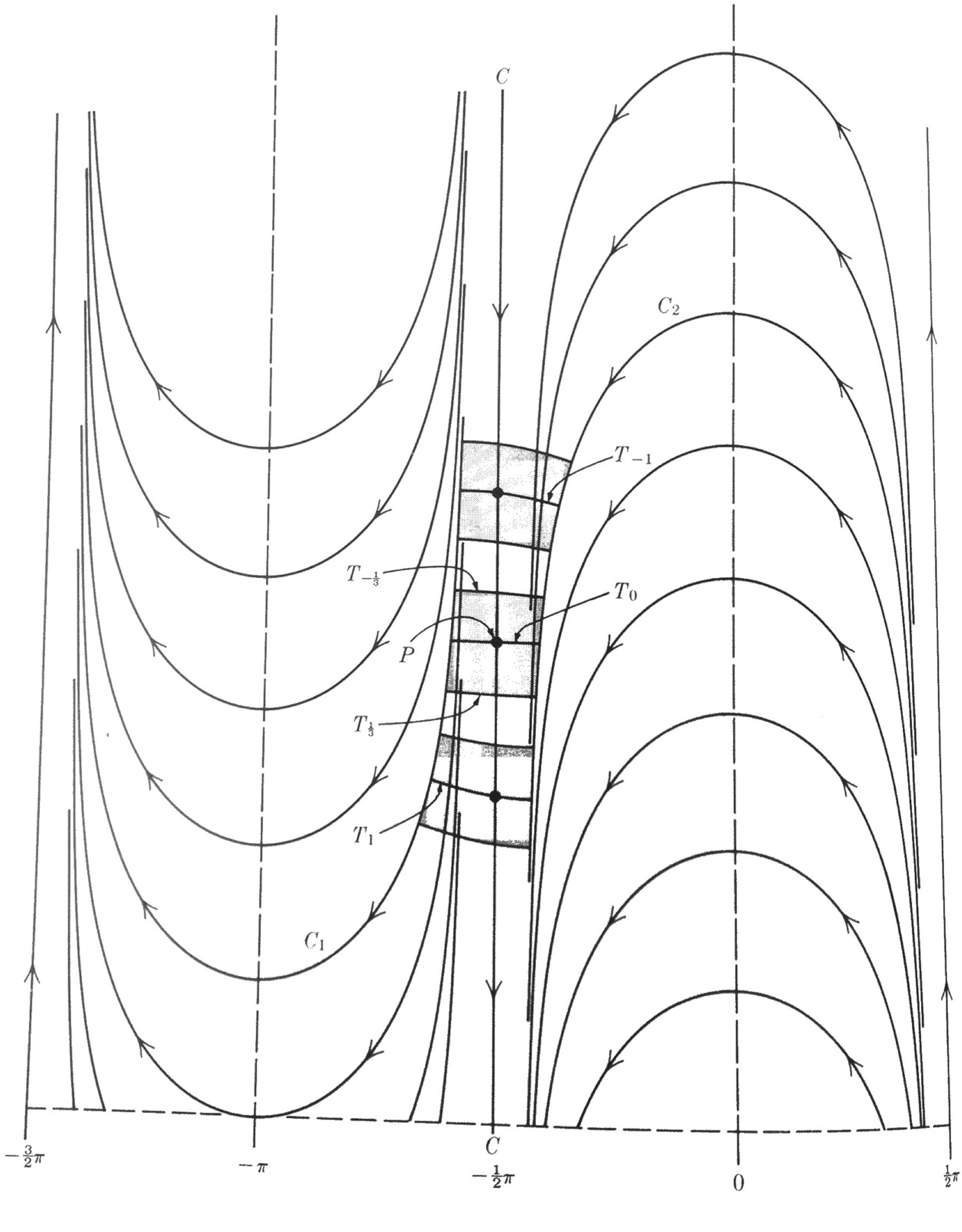

FIGURE 5.2 Diagram for Examples 8.3 and 8.4.

Next, we shall prove that the quotient space is non-Hausdorff. Consider the points

$$P_1 = \left(\frac{\pi}{2}, 0\right), \qquad P_2 = \left(-\frac{\pi}{2}, 0\right)$$

in the plane. We shall prove that their images in the quotient space $\mathbf{R}^2/\mathbf{Z}$ do not have disjoint neighborhoods. For this purpose, it suffices to prove that given *any* neighborhoods N_1 of P_1 and N_2 of P_2, there exists a point in N_1 equivalent to a point of N_2 under the action of the group $\mathbf{Z}$. To do this, consider for any small number $a > 0$ the two points $((\pi/2) - a, 0)$ and $(-(\pi/2) + a, 0)$. These two points are obviously on the same integral curve. How long would it take a particle located at the point $(-(\pi/2) + a, 0)$ to move along its integral curve to the point $((\pi/2) - a, 0)$? To compute this, it obviously suffices to compute how long it would take its projection on the x axis to move from the first position to the second. Because $dx/dt = \cos^2 x$, the time in question is given by the integral

$$I_a = \int_{-(\pi/2)+a}^{(\pi/2)-a} \frac{dx}{\cos^2 x} = \Big[\tan x\Big]_{-(\pi/2)+a}^{(\pi/2)-a}$$

$$= 2 \tan\left(\frac{\pi}{2} - a\right).$$

From this formula for the elapsed time, we can draw several conclusions:

(1) The elapsed time is a continuous function of a.
(2) As $a \to 0$, the elapsed time I_a approaches $+\infty$.
(3) For any number $\varepsilon > 0$, there are infinitely many values of a such that $0 < a < \varepsilon$ and I_a is an integer.

Recall that the points $((\pi/2) - a, 0)$ and $(-(\pi/2) + a, 0)$ are equivalent if and only if the elapsed time I_a is an integer.

From this, the desired conclusion readily follows.

8.4 We next give an example[1] of an infinite cyclic group of homeomorphisms acting without fixed points on a nice space in such a way that the "orbit" of each point is a closed, discrete subspace, but the action is not properly discontinuous! This example shows the strength of the requirement that G be properly discontinuous in Proposition 8.2.

Consider the action of the group of integers $\mathbf{Z}$ on the Euclidean plane $\mathbf{R}^2$ just described. The infinite strip

$$S = \left\{(x, y) : -\frac{\pi}{2} \leqq x \leqq +\frac{\pi}{2}\right\}$$

is mapped into itself by every element of the group $\mathbf{Z}$. We now form a quotient space of S by identifying the points $(\pi/2, y)$ and $(-\pi/2, -y)$ for any real number y. The quotient space is a Möbius strip without boundary (a noncompact sur-

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[1] This example was suggested to the author by Joseph Auslander.

face). Moreover, the action of the group $\mathbf{Z}$ on S is readily seen to be compatible with the identifications, so it also acts on the quotient space. It is clear that the action of $\mathbf{Z}$ on this open Möbius strip is without fixed points, and that the orbit of any point x (i.e., the set of all points $n \cdot x$ for $n \in \mathbf{Z}$) is a discrete, closed subset. The argument given in the last example to show that the quotient space is non-Hausdorff may be applied to this example to show that the point $(\pi/2, 0)$ [which is identified with $(-\pi/2, 0)$] does not have any neighborhood U such that the sets $n \cdot U$ for $n \in \mathbf{Z}$ are pairwise disjoint. Hence, the action of the group on the Möbius strip is not properly discontinuous.

9 Application: The Borsuk-Ulam theorem for the 2-sphere

As usual, let S^n denote the unit n-sphere in $\mathbf{R}^{n+1}$:

$$S^n = \{x \in \mathbf{R}^{n+1} : |x| = 1\}.$$

For any positive integers m and n, let us agree to call a map $f : S^m \to S^n$ *antipode preserving* in case $f(-x) = -f(x)$ for any $x \in S^m$. The following well-known theorem, due to the Polish mathematicians K. Borsuk and S. Ulam, has many interesting consequences.

Theorem 9.1 *There does not exist any continuous, antipode-preserving map $f : S^n \to S^{n-1}$ $(n > 0)$.*

We will prove this theorem only for $n \leqq 2$. Before giving the proof, we indicate and prove some interesting corollaries.

Corollary 9.2 *Assume that $f : S^n \to \mathbf{R}^n$ is a continuous map such that $f(-x) = -f(x)$ for any $x \in S^n$. Then, there exists a point $x \in S^n$ such that $f(x) = 0$.*

PROOF: Assume to the contrary that $f(x) \neq 0$ for all $x \in S^n$. For any $x \in S^n$, define

$$g(x) = \frac{f(x)}{|f(x)|}.$$

Then, g is a continuous map $S^n \to S^{n-1}$, which is antipode preserving, contrary to Theorem 9.1.

Corollary 9.3 *Assume $f : S^n \to \mathbf{R}^n$ is a continuous map. Then, there exists a point $x \in S^n$ such that $f(x) = f(-x)$. In particular, f is not one-to-one.*

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