$$\varphi(\theta,\alpha) = \begin{pmatrix} \sin\theta \cos\alpha \\ \sin\theta \sin\alpha \\ \cos\theta \end{pmatrix}$$

$$\frac{\partial\varphi}{\partial\theta} = \begin{pmatrix} \cos\theta\cos\alpha \\ \cos\theta\sin\alpha \\ -\sin\theta \end{pmatrix} \qquad \frac{\partial\varphi}{\partial\alpha} = \begin{pmatrix} -\sin\theta\sin\alpha \\ \sin\theta\cos\alpha \\ 0 \end{pmatrix}$$

$$g_{ij} = \left\langle \frac{\partial\varphi}{\partial x_i}, \frac{\partial\varphi}{\partial x_j} \right\rangle = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}.$$

$$g^{ij} = (g_{ij})^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2\theta} \end{pmatrix}.$$

$$\Gamma^k_{ij} = \sum_l \frac{1}{2} g^{kl} \left(\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij} \right).$$

$$\Gamma^\theta_{\theta\theta} = \frac{1}{2} g^{\theta\theta} \left(\partial_\theta g_{\theta\theta} \right) = 0$$

$$\Gamma^\theta_{\alpha\theta} = \Gamma^\theta_{\theta\alpha} = \frac{1}{2} g^{\theta\theta} \left(\underbrace{\partial_\theta g_{\theta\alpha}}_{=0} + \underbrace{\partial_\alpha g_{\theta\theta}}_{=0} - \underbrace{\partial_\theta g_{\theta\alpha}}_{=0} \right) = 0$$

$$\Gamma^\theta_{\alpha\alpha} = \frac{1}{2} g^{\theta\theta} \left(-\partial_\theta g_{\alpha\alpha} \right) = \frac{1}{2} \left(-2\sin\theta\cos\theta \right) = -\sin\theta\cos\theta$$

$$\Gamma^\alpha_{\theta\theta} = \frac{1}{2} g^{\alpha\alpha} \left(-\partial_\alpha g_{\theta\theta} \right) = 0$$

$$\Gamma^\alpha_{\theta\alpha} = \Gamma^\alpha_{\alpha\theta} = \frac{1}{2} g^{\alpha\alpha} \left(\partial_\theta g_{\alpha\alpha} + \overbrace{\partial_\alpha g_{\theta\alpha}}^{0} - \overbrace{\partial_\alpha g_{\theta\alpha}}^{0} \right)$$

$$= \frac{1}{2\sin^2\theta} \left(2\sin\theta\cos\theta \right) = \cot\theta.$$

$$\Gamma^\alpha_{\alpha\alpha} = \frac{1}{2} g^{\alpha\alpha} \left(\partial_\alpha g_{\alpha\alpha} \right) = 0.$$

Wir haben das nur

$$\Gamma^\alpha_{\theta\alpha};\ \Gamma^\alpha_{\alpha\theta},\ \Gamma^\theta_{\alpha\alpha} \neq 0.$$

Die Riemannsche Krümmungstensor ist

$$R^l_{ijk} = \partial_i \Gamma^l_{kj} - \partial_j \Gamma^l_{ki} + \sum_m \Gamma^l_{mi}\Gamma^m_{kj} - \Gamma^l_{mj}\Gamma^m_{ki}$$

Wir sehen $R^l_{ijk} = -R^l_{jik}$, sodass $R^l_{iik} = 0$

Wir berechnen

$$R^\theta_{\theta\alpha\theta} = -R^\theta_{\alpha\theta\theta} =$$

$$= \underbrace{\partial_\theta \Gamma^\theta_{\theta\alpha}}_{=0} - \underbrace{\partial_\alpha \Gamma^\theta_{\theta\theta}}_{=0} + \sum_m \underbrace{\Gamma^\theta_{m\theta}}_{=0}\Gamma^m_{\theta\alpha} - \Gamma^\theta_{m\alpha}\underbrace{\Gamma^m_{\theta\theta}}_{=0} = 0$$

$$R^\theta_{\theta\alpha\alpha} = -R^\theta_{\alpha\theta\alpha} = \partial_\theta \Gamma^\theta_{\alpha\alpha} - \underbrace{\partial_\alpha \Gamma^\theta_{\theta\alpha}}_{=0} + \sum_m \underbrace{\Gamma^\theta_{m\theta}}_{=0}\Gamma^m_{\alpha\alpha} - \Gamma^\theta_{m\alpha}\Gamma^m_{\alpha\theta}$$

$$= \partial_\theta \Gamma^\theta_{\alpha\alpha} - \Gamma^\theta_{\alpha\alpha}\Gamma^\alpha_{\alpha\theta} = -\cos^2\theta + \sin^2\theta - \sin\theta\cos\theta\,\frac{\cos\theta}{\sin\theta}$$

$$= \sin^2\theta$$

$$R^\alpha_{\theta\alpha\theta} = -R^\alpha_{\alpha\theta\theta} = \partial_\theta \Gamma^\alpha_{\theta\alpha} - \underbrace{\partial_\alpha \Gamma^\alpha_{\theta\theta}}_{=0} + \sum_m \Gamma^\alpha_{m\theta}\Gamma^m_{\theta\alpha} - \Gamma^\alpha_{m\alpha}\underbrace{\Gamma^m_{\theta\theta}}_{=0}$$

$$= \partial_\theta \Gamma^\alpha_{\theta\alpha} + \Gamma^\alpha_{\alpha\theta}\Gamma^\alpha_{\theta\alpha} = -\frac{1}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = 1$$

$$R^\alpha_{\theta\alpha\alpha} = -R^\alpha_{\alpha\theta\alpha} = \partial_\theta \underbrace{\Gamma^\alpha_{\alpha\alpha}}_{=0} - \underbrace{\partial_\alpha \Gamma^\alpha_{\alpha\theta}}_{=0} + \sum_m \Gamma^\alpha_{m\theta}\Gamma^m_{\alpha\alpha} - \Gamma^\alpha_{m\alpha}\Gamma^m_{\alpha\theta}$$

$= 0$. Alle andere Komponenten verschwinden auch

$$v = \frac{\partial \varphi}{\partial \theta} \qquad w = \frac{1}{\sin^2\theta}\frac{\partial \varphi}{\partial \alpha}$$

is ein orthonormale basis.

$$\cancel{R_p(v,w)w = \frac{R^{\theta}{}_{\theta\alpha\alpha}}{(\sin^2\theta)^2}\frac{\partial \varphi}{\partial \theta} = \frac{1}{\sin^2\theta}\frac{\partial \varphi}{\partial \theta}}$$

$$R_p(v,w)w = \frac{1}{\sin^2\theta} R^{\theta}{}_{\theta\alpha\alpha} \frac{\partial \varphi}{\partial \theta} = \frac{\partial \varphi}{\partial \theta}.$$

So dass $\quad I(R_p(v,w)w, v) = I\left(\frac{\partial \varphi}{\partial \theta}, \frac{\partial \varphi}{\partial \theta}\right) = 1.$

MB $12^2/12$

Exercise 2.:

This follows from Lemma 4.3.4.

As $R(v,w)w = K(p)\big(I(w,w)v - I(v,w)w\big)$

we find $I(R(v,w)w, v)$

$$= K(p)\big(I(w,w)I(v,v) - I(v,w)I(w,v)\big). \qquad \square.$$

Exercise 3:

$$\cancel{R_{mi,jk} = \sum_{l=1}^{2} g_{ml} R^{l}_{ijk}.}$$

$$\cancel{= \partial_i \Gamma \ldots g_{ki}}$$

This can be done from definitions, ~~or~~:

$R_{mijk} + R_{mjki} + R_{mkij} =$

$$= \sum_{l=1}^{2} g_{ml}\Big\{\big(\partial_i \Gamma^{l}_{kj} - \partial_j \Gamma^{l}_{ki}\big) + \sum_n \Gamma^{l}_{ni}\Gamma^{n}_{kj} - \Gamma^{l}_{nj}\Gamma^{n}_{ki}\Big\} +$$

$$\sum_{l=1}^{2} g_{ml}\Big\{\big(\partial_j \Gamma^{l}_{ik} - \partial_k \Gamma^{l}_{ij}\big) + \sum_n \Gamma^{l}_{nj}\Gamma^{n}_{ik} - \Gamma^{l}_{nk}\Gamma^{n}_{ij}\Big\} +$$

$$\sum_{l=1}^{2} g_{ml}\Big\{\big(\partial_k \Gamma^{l}_{ji} - \partial_i \Gamma^{l}_{jk}\big) + \sum_n \Gamma^{l}_{nk}\Gamma^{n}_{ji} - \Gamma^{l}_{ni}\Gamma^{n}_{jk}\Big\}.$$

$= 0$ (Same symbol drops out)

Follows of ~~cou~~ course also from 4.3.3.