Mathematisches Institut Universität zu Köln
Prof. Dr. Sabatini October 29th 2014

Algebraic Topology – Homework 3

Due date : November 5th in class
Mandatory exercises : 1, 2, 3 and 4. Optional : 5

Exercise 1.

(a) Let X be a topological space and $x_0 \in X$. Prove that if X is endowed with the discrete topology (i.e. $\mathcal{T} = \mathcal{P}(X)$, the so called power set of X) then $\pi_1(X, x_0) = \{[c_{x_0}]\}$, where c_{x_0} is the constant loop.
(b) Calculate $\pi_1(\mathbb{Q}, 0)$, where $\mathbb{Q}$ is the set of rationals in $\mathbb{R}$, endowed with the subset topology ($\mathbb{R}$ has the standard Euclidean topology).

Exercise 2.

Exercise 16 (a), (b), (c) and (f) on page 39, Chapter 1 of Hatcher's book, available online here. (In (f) the author actually takes the Möbius band as being the closure of the open surface we defined in class; the boundary component you need to add is precisely what he calls A.)

Exercise 3.

(a) Prove that two continuous paths $\alpha, \beta \colon I \longrightarrow X$ such that $\alpha(0) = \beta(0) = x$ and $\alpha(1) = \beta(1) = y$, give rise to the same isomorphism from $\pi_1(X, x)$ to $\pi_1(X, y)$ if and only if the homotopy class of the loop $[\beta * \overline{\alpha}]$ is in the center [1] of $\pi_1(X, x)$.
(b) Let π_α be the isomorphism between $\pi_1(X, x)$ and $\pi_1(X, y)$ determined by the path $\alpha \colon I \longrightarrow X$, with $\alpha(0) = x$ and $\alpha(1) = y$. Prove that π_α is independent of α if and only if $\pi_1(X, x)$ is abelian.

1. Recall that the center $Z(G)$ of a group G is defined to be $Z(G) = \{a \in G \mid a \cdot b = b \cdot a, \ \forall b \in G\}$

Exercise 4.

This exercise is devoted to proving the following

Theorem 1. *The fundamental group of a path-connected topological group* $(G, \star)$ *is abelian.*

In order to prove it, you can follow the next steps. Let $f, g \in \mathcal{L}_e$, the space of loops at $e \in G$, where e denote the identity element in G. Recall that we defined the "product of two paths" as their juxtaposition $(f * g)(s) = f(2s)$ for $s \in [0, \frac{1}{2}]$ and $(f * g)(s) = g(2s - 1)$ for $s \in [\frac{1}{2}, 1]$. Since G is also a group, we can also define another product of paths as $(f \star g)(s) = f(s) \star g(s)$, for all $s \in [0, 1]$.

(a) Prove that the product of paths $\star$ on $\mathcal{L}_e$ passes to homotopy classes of loops, and that $[f][g] = [f] \star [g]$. (Hint : observe that $f * g = (f * c_e) \star (c_e * g)$, where c_e is the constant loop at e.)

Note that you can define the path $f^{-1}(s) := (f(s))^{-1}$, for all $s \in [0, 1]$. So from (a) you have $[f]^{-1}(s) = [(f(s))^{-1}]$ for $s \in [0, 1]$.

(b) Construct a homotopy between $f \star g$ and $g \star f$. (Hint : what happens if you multiply $(f \star g)(s)$ on the left by a path $h_1(st)$ which is e for $t = 0$ and $f^{-1}(t)$ for $s = 1$, and by a path $h_2(st)$ which is e for $t = 0$ and $f(t)$ for $s = 1$?)

Exercise 5.

Let X be a topological space which is connected and locally path-connected. Prove that X is path connected.