Mathematisches Institut | Universität zu Köln
Prof. Dr. Sabatini | 01.04.2019

Algebraic Topology – Homework 0

No due date

Exercise 1.

(a) Prove that, if $(X, \mathcal{T})$ is a Hausdorff space and $C \subseteq X$ is a compact subset, then C is closed.

(b) Using (a), prove that if $\pi\colon (P, \mathcal{T}_1) \longrightarrow (S, \mathcal{T}_2)$ is a continuous surjective map, with $(P, \mathcal{T}_1)$ compact and $(S, \mathcal{T}_2)$ Hausdorff, then $\mathcal{T}_2$ coincides with the quotient topology.

The next exercise is a "Prove or disprove exercise" : it means that, if you "feel" that the assertion is true, you should prove it carefully ; if you "feel" it is false, you should find a contradiction. This type of exercises are harder but very instructive, as they improve intuition.
Here every subspace of $(\mathbb{R}^n, \mathcal{E})$ is endowed with the subset topology, and $\mathcal{E}$ denotes the Euclidean topology on $\mathbb{R}^n$,

Let $S^n \subset \mathbb{R}^{n+1}$ be the n-dimensional unit sphere

$$S^n := \left\{ (x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = 1 \right\},$$

and let $\sim$ be the equivalence relation on S^1 defined by :

$$(x_1, x_2, \ldots, x_{n+1}) \sim (y_1, y_2, \ldots, y_{n+1}) \iff (x_1, x_2, \ldots, x_{n+1}) = \pm(y_1, y_2, \ldots, y_{n+1}).$$

Then $(\mathbb{R}\mathrm{P}^n, \mathcal{T})$ is the topological space given by the equivalence classes of $\sim$, endowed with the quotient topology (with respect to the map that sends $(x_1, x_2, \ldots, x_{n+1}) \in S^n$ to $[(x_1, x_2, \ldots, x_{n+1})] = \{(x_1, x_2, \ldots, x_{n+1}), (-x_1, -x_2, \ldots, -x_{n+1})\}$), and is called the **$n$-dimensional (real) projective space**. In other words, "$\mathbb{R}\mathrm{P}^n$ is obtained from S^n by identifying antipodal points."

Exercise 2.

Prove or disprove* the existence of the following homeomorphisms :

(a) $\mathbb{R}\mathrm{P}^n \approx S^n$ for all integer $n \geq 0$;

(b) $S^1 \approx \infty$ (here "∞" is the subspace of $\mathbb{R}^2$ given by two circles touching in one point, as in the infinity symbol) ;

(c) $\mathbb{R}^1 \approx \mathbb{R}^n$, for all integers $n > 1$;

(d) $\mathbb{R}^2 \approx \mathbb{R}^n$, for all integers $n > 2$;

(e) $S^2 \setminus \{\mathbf{n}, \mathbf{s}\} \approx \mathcal{C}$, where S^2 is the unit sphere $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$, $\mathbf{n} = (0, 0, 1)$, $\mathbf{s} = (0, 0, -1)$ and $\mathcal{C}$ is the cylinder given by $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$.

(e) $\mathbb{R}^{n+1} \setminus \{0\} \approx S^n \times \mathbb{R}$, where S^n is the n-dimensional unit sphere.

* Here *Prove* means that you should find a homeomorphism *as explicitly as possible.*

Exercise 3.

Prove that every injective continuous map $S^1 \longrightarrow S^1$ is also surjective.

Exercise 4.

For an integer $n \geq 1$, is every map $S^n \longrightarrow S^1$ null-homotopic ?